Math 10A
Worksheet, Discussion #31; Monday, 7/30/2018
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1 Matrix Algebra

1.1 Concepts

1. A **matrix** is a $m \times n$ grid of numbers. This mean m rows and n columns. A **vector** can either be a **row vector** or **column vector**. A **row vector** is just a single row, so a $1 \times n$ matrix and a **column vector** is a column or a $m \times 1$ matrix. A **scalar** is just a number.

   We can add two matrices if they are of the same size. We add each entry separately. We can also multiply matrices by scalars.

   Given two matrices A, B that are of dimension $m \times n$ and $\ell \times k$, we can multiply them as AB if and only if $n = \ell$. So the number of columns in the first matrix must equal the number of rows in the second matrix. This means that sometimes we can compute AB but not BA. If you multiply a $m \times n$ matrix by a $n \times k$ matrix, the outcome is a $m \times k$ matrix.

   If two vectors have the same number of elements, then we can take the dot product of them. The **dot product** of the vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ is a scalar given by $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. We write it as $\vec{v} \bullet \vec{w}$. The **norm** of a vector is given by $\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ and denoted by $|\vec{v}|$. Then $|a| = 0$ if and only if $a = 0$, the 0 vector. Let θ be the angle between two vector $\vec{v}, \vec{w}$, then $\vec{v} \bullet \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos\theta$.

   For a $m \times n$ matrix A, the **transpose** A^T is the $n \times m$ matrix with all the elements flipped around.

1.2 Problems

2. True **FALSE** If A is a matrix and v is a vector, then Av (assuming we can take such a product) is another vector.

   Solution: If we multiply a 5×1 matrix with a 1×5 row vector, then the output is a 5×5 matrix that is not a vector. But, if v is a column vector, namely if v is 5×1, then the answer is true.

3. True **FALSE** If there are matrices such that $AB = M$ and we know the dimensions of M, then we know the dimensions of A and B.

Solution: A $(2 \times 2)(2 \times 2) = (2 \times 2)$ and $(2 \times 3)(3 \times 2) = (2 \times 2)$ so we need know the inside dimension as well.

4. Let $A = \begin{pmatrix} 3 & 5 & 6 \\ 1 & 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 3 & 3 \\ 0 & -2 \end{pmatrix}$. Calculate $A + 2B^T$ and AB.

Solution:

$$A + 2B^T = \begin{pmatrix} 1 & 11 & 6 \\ 1 & 8 & -2 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 12 & 3 \\ 5 & 2 \end{pmatrix}$$

5. Let $A = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -1 \\ -3 & -4 \\ 0 & 1 \end{pmatrix}$. Calculate AB and BA.

Solution:

$$AB = \begin{pmatrix} 1 & -11 \\ -5 & 5 \end{pmatrix} \qquad BA = \begin{pmatrix} 11 & 15 & 11 \\ -2 & -9 & -25 \\ -1 & 0 & 4 \end{pmatrix}.$$

6. Represent the system of equations $3x + 5y + 2z = 11, 8x - y = 0$ in matrix form as $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$ with A being a matrix and b a vector.

Solution:

$$\begin{pmatrix} 3 & 5 & 2 \\ 8 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}.$$

7. Let $v = (1, 2, 2, -1)$ and $w = (5, 3, -5, 3)$. Calculate $v \bullet w$ and $|v|$.

Solution: $v \bullet w = 1 \cdot 5 + 2 \cdot 3 + 2 \cdot (-5) + (-1) \cdot 3 = -2$. $|v| = \sqrt{1^2 + 2^2 + 2^2 + (-1)^2} = \sqrt{10}$.

8. Find the angle between the two vector $v = (1, 3, 5, -2, 4, 3)$ and $w = (1, 1, 5, 2, 2, 1)$.

Solution: If θ is the angle between them, then

$$\cos\theta = \frac{v \bullet w}{|v| \cdot |w|} = \frac{36}{\sqrt{64} \cdot \sqrt{36}} = \frac{36}{8 \cdot 6} = \frac{3}{4}.$$

Thus $\theta = \arccos(3/4) \approx 0.7227$.

9. Suppose that A is a matrix such that $A \begin{pmatrix} 1 \\ 1 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$. What are the dimensions of A? Come up with an example for A. Is the size unique? Is A unique?

Solution: Let A be a $m \times n$ matrix. Then $(m \times n) \cdot (3 \times 1) = (2 \times 1)$. Thus $m = 2$ and $n = 3$ and this size is unique. But there are many choices for A, one such choice is

$$A = \begin{pmatrix} 3 & 8 & -1 \\ 5 & 0 & 0 \end{pmatrix}$$

10. When is $|\vec{v} \bullet \vec{w}| = |\vec{v}| \cdot |\vec{w}|$? (Hint: What is θ?)

Solution: We know that $|\vec{v} \bullet \vec{w}| = ||v| \cdot |w| \cdot \cos\theta| = |v| \cdot |w| \cdot |\cos\theta|$. Thus $|\cos\theta| = 1$ and hence $\alpha = 0, \pi$. Therefore, the vectors must on the same line.

11. Find a 2×2 matrix A with no 0's such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Solution: One choice is $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

12. Find x, y such that

$$\begin{pmatrix} -3 & 2 \\ y & x \end{pmatrix} \begin{pmatrix} -2 & 5 \\ x & y \end{pmatrix} = \begin{pmatrix} y & -7 \\ -7 & 16 \end{pmatrix}$$

Solution: Multiplying gives

$$\begin{pmatrix} 6+2x & 2y-15 \\ x^2-2y & xy+5y \end{pmatrix} = \begin{pmatrix} y & -7 \\ -7 & 16 \end{pmatrix}.$$

Thus $2y - 15 = -7$ and hence $2y = 8$ or $y = 4$. Then we have that $2x + 6 = y = 4$ so $2x = 4 - 6 = -2$ and $x = -1$.